

# Solution for the dispersive and dissipative atom-field Hamiltonian under time dependent linear amplification processes

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## Abstract

The dispersive interaction between a two-level atom and a quantized field is studied. We consider besides a time dependent linear amplification and dissipative processes. In order to solve the master equation for this system, we use superoperator techniques.

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## I. INTRODUCTION

Recently, the geometric phase due to the Stark shift in a system composed of a field, driven by time-dependent linear amplification, interacting dispersively with a two-level (fermionic) system was studied [1]. The solution for the Hamiltonian that takes into account the above conditions, due to its time dependence, was solved by using invariant techniques of the Lewis-Ermakov type [2, 3]. It is well known that dissipative dynamics must be considered when atoms interact with fields in the dispersive regime (far off-resonance) as the atom-field interaction constant is replaced by a much smaller *dispersive* interaction constant.

In this contribution we want to study the effect to add the interaction with the environment for this system. we will do this by expressing the master equation related to each element of the density matrix and making a transformation that allows a solution to the appropriate differential equation.

## II. DISPERSIVE INTERACTION

Consider the two-level atom-field interaction Hamiltonian

$$H_{a-f} = \omega a^\dagger a + \frac{\omega_0}{2} \sigma_z + \lambda (a \sigma_+ + a^\dagger \sigma_-), \quad (1)$$

where  $\lambda$  is the atom-field interaction constant,  $\omega_0$  is the atomic transition frequency and  $\sigma_-$  ( $\sigma_+$ ) is the lowering (raising) operator for the atom, with  $[\sigma_+, \sigma_-] = 2\sigma_z$  and  $a$  and  $a^\dagger$  are the annihilation and creation operators for the cavity field, respectively. The field frequency is  $\omega$ . Note that this interaction may also be realized in the ion-laser interaction [4, 5].

If we consider the field frequency far away from the atomic transition frequency, i.e.  $|\omega - \omega_0| \geq \lambda$ , the atom and the field stop exchanging energy and the Hamiltonian above can be cast into a dispersive Hamiltonian either by using adiabatic [6] or small rotation techniques [7]. The Hamiltonian is then written as

$$H_{eff} = \nu a^\dagger a + \frac{\omega}{2} \sigma_z + \chi a^\dagger a \sigma_z + \chi \sigma_{ee}, \quad (2)$$

with  $\sigma_{ee} = \frac{1}{2}(1 - \sigma_z)$ . Now we consider the effective Hamiltonian for the dispersive interaction between a two-level atom and a quantized field under time dependent linear amplification process [1]

$$H = \nu a^\dagger a + \frac{\omega}{2} \sigma_z + \chi a^\dagger a \sigma_z + \chi \sigma_{ee} + f(t) a^\dagger + f^*(t) a. \quad (3)$$

The von Neumann equation for the density matrix  $\mathcal{R}$  taking into account the environment is

$$\dot{\varrho} = -i[H, \varrho] + \mathcal{L}\varrho, \quad (4)$$

where [8]

$$\mathcal{L}\rho = \gamma(J - L)\rho, \quad (5)$$

with

$$L\rho = a^\dagger a \rho + \rho a^\dagger a, \quad J\rho = 2a\rho a^\dagger. \quad (6)$$

Duzzioni *et al.* [1] studied the Berry phase by solving the Schrödinger equation for the Hamiltonian (3) by using Lewis-Ermakov techniques [2], commonly used to solve time dependent harmonic oscillator interactions [3]. Here we show how to solve this interaction with a different method: a simple transformation that allows solution even when losses are taken into account.

### III. SOLUTION TO THE MASTER EQUATION

We can simplify the master equation by transforming it via  $\varrho = \exp[-it(\frac{\omega}{2}\sigma_z + \chi\sigma_{ee} + \nu a^\dagger a)]\rho \exp[it(\frac{\omega}{2}\sigma_z + \chi\sigma_{ee} + \nu a^\dagger a)]$  such that we obtain

$$H_T = \chi a^\dagger a \sigma_z + f_\nu(t) a^\dagger + f_\nu^*(t) a, \quad (7)$$

with  $f_\nu(t) = f(t)e^{i\nu t}$ .

$$\dot{\rho} = -i[H_T, \rho] + \mathcal{L}\rho \quad (8)$$

Writing the master equation for each element of the density matrix we have

$$\dot{\rho}_{ee} = [R + S(f_\nu) + \mathcal{L}]\rho_{ee}, \quad (9)$$

$$\dot{\rho}_{gg} = [-R + S(f_\nu) + \mathcal{L}]\rho_{gg}, \quad (10)$$

$$\dot{\rho}_{eg} = [S(f_\nu) + \mathcal{L} - i\chi L]\rho_{eg}, \quad (11)$$

and

$$\dot{\rho}_{ge} = [S(f_\nu) + \mathcal{L} + i\chi L]\rho_{ge}, \quad (12)$$

with

$$R\rho = -i\chi a^\dagger a \rho + i\rho \chi a^\dagger a, \quad (13)$$

and

$$S(f_\nu)\rho = -i[f_\nu(t)a^\dagger + f_\nu^*(t)a]\rho + i\rho[f_\nu(t)a^\dagger + f_\nu^*(t)a]. \quad (14)$$

Note that the relevant commutators are

$$[S(\epsilon), \mathcal{L}]\rho = S(\epsilon)\rho, \quad (15)$$

$$[J, L]\rho = 2J\rho, \quad (16)$$

and

$$[R, J]\rho = [R, L]\rho = 0. \quad (17)$$

#### A. Solution for $\rho_{ee}$

We first transform (9) with  $\rho_{ee} = \exp\{(R + \mathcal{L})t\}\tilde{\rho}_{ee}$  to obtain

$$\dot{\tilde{\rho}}_{ee} = e^{\gamma t} S(f_{\nu+\chi})\tilde{\rho}_{ee} = -i [[g_+(t)a^\dagger + g_+^*(t)a], \tilde{\rho}_{ee}] \quad (18)$$

with  $g_+(t) = f(t)e^{i(\nu+\chi)t+\gamma t}$  with solution

$$\tilde{\rho}_{ee}(t) = D^\dagger[iG_+(t)]\tilde{\rho}_{ee}(0)D[iG_+(t)] \quad (19)$$

with  $G_+(t) = \int_0^t g_+(t)dt$  and  $D(\beta) = e^{\beta a^\dagger - \beta^* a}$  the Glauber displacement operator [9].

#### B. Solution for $\rho_{gg}$

We follow the solution for  $\rho_{ee}$  above and transform (10) with  $\rho_{gg} = \exp\{(-R + \mathcal{L})t\}\tilde{\rho}_{gg}$  to obtain

$$\dot{\tilde{\rho}}_{gg} - i [[g_-(t)a^\dagger + g_-^*(t)a], \tilde{\rho}_{gg}] \quad (20)$$

with  $g_-(t) = f(t)e^{i(\nu-\chi)t+\gamma t}$  with solution

$$\tilde{\rho}_{gg}(t) = D^\dagger[iG_-(t)]\tilde{\rho}_{gg}(0)D[iG_-(t)] \quad (21)$$

with  $G_-(t) = \int_0^t g_-(t)dt$ .

### C. Solution for $\rho_{eg}$ and $\rho_{ge}$

The solution for  $\rho_{eg}$  (or  $\rho_{ge}$ ) is more complicated because it involves non Hermitian operators. Several straightforward (but tedious) transformation to simplify the equation may be performed. We start with  $\rho_{eg} = \exp\left(-\frac{\gamma}{2\beta}J\right)\rho_{eg}^{(1)}$  with  $\beta = \gamma + i\chi$  to obtain

$$\dot{\rho}_{eg}^{(1)} = [S(f_\nu) + \frac{\gamma}{2\beta}S_1 - \beta L]\rho_{eg}^{(1)} \quad (22)$$

with

$$S_1\rho = -2i(f_\nu\rho a^\dagger - f_\nu^*a\rho). \quad (23)$$

By applying

$$\rho_{eg}^{(2)} = e^{\beta ta^\dagger a}\rho_{eg}^{(1)}e^{\beta ta^\dagger a} \quad (24)$$

we end up with the equation

$$\dot{\rho}_{eg}^{(2)} = -i[F_1(\beta)a^\dagger + F_2(\beta)a] + i\rho_{eg}^{(2)}[F_3(\beta)a^\dagger + F_4(\beta)a] \quad (25)$$

with  $F_1(\beta) = f_\nu e^{\beta t}$ ,  $F_2(\beta) = f_\nu^* e^{-\beta t}(1 - \gamma/\beta)$ ,  $F_3(\beta) = f_\nu e^{-\beta t}(1 - \gamma/\beta)$  and  $F_4(\beta) = f_\nu^* e^{\beta t}$ . This equation looks now easy to integrate. With  $G_j(\beta, t) = \int F_j(\beta)dt$  we write the solution to the above equation as

$$\begin{aligned} \rho_{eg}^{(2)}(t) &= e^{-\int_0^t (F_2(\beta)G_1(\beta) + F_3(\beta)G_4(\beta))dt} \times \\ &e^{-iG_1(\beta)a^\dagger} e^{-i[G_2(\beta) - G_2(\beta;0)]a} e^{iG_1(\beta;0)a^\dagger} \rho_{eg}^{(2)}(0) \\ &e^{-iG_4(\beta;0)a} e^{i[G_3(\beta) - G_3(\beta;0)]a^\dagger} e^{iG_4(\beta)a} \end{aligned} \quad (26)$$

and

$$\begin{aligned} \rho_{eg}^{(1)}(t) &= e^{-\int_0^t (F_2(\beta)G_1(\beta) + F_3(\beta)G_4(\beta))dt} \times \\ &e^{-\beta ta^\dagger a} e^{-iG_1(\beta)a^\dagger} e^{-i[G_2(\beta) - G_2(\beta;0)]a} e^{iG_1(\beta;0)a^\dagger} \rho_{eg}^{(1)}(0) \\ &e^{-iG_4(\beta;0)a} e^{i[G_3(\beta) - G_3(\beta;0)]a^\dagger} e^{iG_4(\beta)a} e^{-\beta ta^\dagger a} \end{aligned} \quad (27)$$

The solution for  $\rho_{ge}$  is similar to the one for  $\rho_{eg}$  but taking  $\beta \rightarrow \beta^*$ .

## IV. CONCLUSIONS

We have studied the dispersive interaction between a two-level atom and an electromagnetic field in the presence of dissipation and time dependent linear amplification processes.

By transforming the master equation [10] we have managed to produce simpler master equations for each element of the density matrix, which we have shown to be solvable. Systems like the ones studied here are of interest in the reconstruction of quasiprobability distribution functions to measure the quantum state of light [11, 12].

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